Various Statistical Measures of a Parallel Redundant Complex System with Two Types of Failure

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Abstract
In this paper we analyze a Markov model consisting of two non-identical units in parallel redundancy. System suffers failure in two models; viz., normal mode (N.M) and common case (C.C). The repair, wherever undertaken follows general distribution. After the failure of one of the two units, the failure rate of another unit is assumed to be increased as compared to the situation on while it works jointly. This is termed as over loading effect.

Keywords: Statistical Measures, Redundant Complex System, Types of Failure.

1. Introduction

As the field of reliability engineering is becoming a recognized discipline in engineering, so is the awareness of its specialized topics which generally overlooked in the past. For example, recent years common cause failures have received a widespread attention in reliability analysis of redundant components, units or systems, because the assumption of statistical-independent failure of redundant units is easily violated in practice.

A common-cause failure is defined as any instance where multiple units or components fail due to a single cause.

In the recent past, several researchers [1, 2, 3, 4, 5] have contributed a lot in reliability field while analyzing various complex system mathematically, incorporating the concept of common-cause failure (may occur due to equipment design deficiency, operation and maintenance error, external environment, external catastrophe and function deficiency etc.).

B.S. Dhillon and H.C. Viswanath [34] presented three models with common-cause failure and studied their reliability behavior. They considered a parallel redundant system consisting of two non-identical units assuming that the system or any single unit may collapse either due to normal failure or by the common-cause failure, they also assumed that repair and failure both follow exponential time distributions. The additional assumption in their analysis was that failure rate of either of the two units remains invariably the same whether it operates alone or jointly. The assumptions made by the earlier researchers are not realistic to practical situation problems as while one unit of the complex system fails. The failure rate of the other unit must increase positively due to the over-loading effect. Not only this, constant repair of the unit/system leads to wastage of time and cost both.

Keeping these facts in view, the author in this paper has therefore, analyzed a Markova’s model consisting of two non-identical units in parallel redundancy. System suffers failure in two models; viz., normal mode (N.M) and common case (C.C). The repair, wherever undertaken follows general distribution. After the failure of one of the two units, the failure rate of another unit is assumed to be increased as compared to the situation on while it works jointly. This is termed as over loading effect.

Several reliability measures; viz., L.T. of various state probabilities, up and down state
probabilities, M.T.T.F. (mean time to failure), variance, reliability of the system etc. have been computed using supplementary and L.T. techniques. To connect the utility of the model with practical life consideration, some graphs have also been appended at the end.

2. Assumptions

1. Initially, the system is good;
2. Units are non-identical and active;
3. A common-cause failure occurs at any of the operable states;
4. All failure rates are exponential whereas repair rates are arbitrary;
5. System is in failed state when both the units have failed;
6. Repair facility is available when either one unit or both the units have failed; and
7. After repair, system/unit works like a new.

STATE TRANSITION DIAGRAM

FIG.1
3. Notations

S_o,o Good state of the system, when both the units are active;
S_i,o Good state of the system, when first unit has failed and second is active;
S_o,f Good state of the system, when second units has failed and first is active;
S_i,f Failed state of the system when both the units have failed due to N,M;
S_o,c Failed state of the system due to C,C;
\( \lambda_1 \) Failure rate of first/second unit when operating jointly;
\( \lambda_1 \) Failure rate of first/second unit when operating alone;
\( C_1 \) Common-cause failure rate of the system when both units are good;
\( C_2 \) Common-cause failure rate of the system when first/second unit is active;
\( \mu_1(x) \) Transition repair rate from state S_o,o/ 
\( \mu_2(y) \) S_o,o/S_i,o/S_o,c to S_o,o;
\( P_{i,j}(x,t) \) / \( P_{i,j}(y,t) \) General pdf [system is in state \( S_{i,j} \) and \( P_{i,j}(u,t) \) is under repair; elapsed repair time is \( x/y/z/u/t \)]
[(i,j) means (f,o), (o,f) (f,f), (c,c)];
\( f(s) \) Laplace-transform of the function \( f(t) \);
\( S \) Laplace-transform variable;
\( \int \) Integrated notation \( (o,\infty) \), unless otherwise stated;
\( S_i(x) = \mu_i(x) \exp \left[ - \int_0^x \lambda_i(x)dx \right] \);
\( M_i \) Mean time to repair.

4. Formulation of Mathematical Model

Elementary probability consideration and continuity arguments yield the following difference-differential equations associated with fig.1

\[
\frac{d}{dt} + \lambda_1 + \lambda_2 + \lambda_{c1} \int_0^x P_{i,o}(t) = \int_0^x P_{f,o}(x,t)\mu_1(x)dx + \int_0^y P_{o,f}(y,t)\mu_2(y)dy + \\
\int_0^z P_{f,z}(z,t)\mu_3(z)dz + \int_0^u P_{o,c}(u,t)\mu_i(u)du
\]

\[
\frac{d}{dx} + \lambda_1 + \lambda_2 + \mu_i(x) \int_0^x P_{i,o}(t) = 0
\]
5. Solution of the Model

Taking Laplace-transforms of equations (1-9) and using equation (10), one may obtain:

\[
\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty P_f,0(0,t) = \lambda_1 P_0,0(t) \\
P_0, f(0,t) = \lambda_2 P_0,0(t) \\
P_f, f(0,t) = \lambda_3 P_f,0(t) + \lambda_1^2 P_0, f(t) \\
P_c, c(0,t) = \lambda_4 P_c,0(t) + \lambda_3 P_f,0(t) + \lambda_2 P_0, f(t) \\
\]

Initial Conditions

\[
P_f,0(0) = 1 \\
P_0, f(0) = P_t, f(0) = P_c, c(0) = 0 \\
\]

\[
S + \lambda_1 + \lambda_2 + \lambda_3 \int P_o, o(s) = 1 + \int_0^\infty P_f, o(x,s) \mu_1(x) dx + \int_0^\infty P_c, f(y,s) \mu_2(y) dy + \\
\int_0^\infty P_f, z(s) \mu_3(z) dz + \int_0^\infty P_c, u(s) \mu_4(u) du \\
\]

Boundary Conditions

\[
P_f,0(0,t) = \lambda_4 P_0,0(t) \\
P_0, f(0,t) = \lambda_2 P_0,0(t) \\
P_f, f(0,t) = \lambda_3 P_f,0(t) + \lambda_1^2 P_0, f(t) \\
P_c, c(0,t) = \lambda_4 P_c,0(t) + \lambda_3 P_f,0(t) + \lambda_2 P_0, f(t) \\
\]

\[
S + \lambda_1 + \lambda_2 + \lambda_3 \int P_o, o(s) = 1 + \int_0^\infty P_f, o(x,s) \mu_1(x) dx + \int_0^\infty P_c, f(y,s) \mu_2(y) dy + \\
\int_0^\infty P_f, z(s) \mu_3(z) dz + \int_0^\infty P_c, u(s) \mu_4(u) du \\
\]

\[
S + \lambda_1 + \lambda_2 + \lambda_3 \int P_o, o(s) = 1 + \int_0^\infty P_f, o(x,s) \mu_1(x) dx + \int_0^\infty P_c, f(y,s) \mu_2(y) dy + \\
\int_0^\infty P_f, z(s) \mu_3(z) dz + \int_0^\infty P_c, u(s) \mu_4(u) du \\
\]

\[
S + \lambda_1 + \lambda_2 + \lambda_3 \int P_o, o(s) = 1 + \int_0^\infty P_f, o(x,s) \mu_1(x) dx + \int_0^\infty P_c, f(y,s) \mu_2(y) dy + \\
\int_0^\infty P_f, z(s) \mu_3(z) dz + \int_0^\infty P_c, u(s) \mu_4(u) du \\
\]

\[
S + \lambda_1 + \lambda_2 + \lambda_3 \int P_o, o(s) = 1 + \int_0^\infty P_f, o(x,s) \mu_1(x) dx + \int_0^\infty P_c, f(y,s) \mu_2(y) dy + \\
\int_0^\infty P_f, z(s) \mu_3(z) dz + \int_0^\infty P_c, u(s) \mu_4(u) du \\
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\[
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\int_0^\infty P_f, z(s) \mu_3(z) dz + \int_0^\infty P_c, u(s) \mu_4(u) du \\
\]
6. Calculation of $P_i(S)$

In view of equation (16-19), equations (11-15) yield the following Laplace-transforms of the system's state probabilities:

$$\tilde{P}_{0,0}(s) = \frac{1}{A(s)}$$

(20)

$$\tilde{P}_{f,0}(s) = \lambda_i z_1(s + \lambda_2 + \lambda_i) \frac{1}{A(s)}$$

(21)

$$\tilde{P}_{0,f}(s) = \lambda_2^i z_2(s + \lambda_i^i + \lambda_i) \frac{1}{A(s)}$$

(22)

$$\tilde{P}_{f,f}(s) = \left[\lambda_i^i \lambda_2^i z_1(s + \lambda_2^i + \lambda_i) + \lambda_i \lambda_2 z_2(s + \lambda_i + \lambda_i)\right] \frac{Z_i(s)}{A(s)}$$

(23)

$$\tilde{P}_{c,c}(s) = \left[\lambda_i^i \lambda_2^i z_1(s + \lambda_2^i + \lambda_i) + \lambda_2^i \lambda_i z_2(s + \lambda_i + \lambda_i)\right] \frac{Z_i(s)}{A(s)}$$

(24)

Also, L.T. of the probability that the system is in operable state at time $t$ is given by

$$\tilde{P}_{up}(s) = \left[1 + \lambda_i^i \lambda_2^i z_1(s + \lambda_2^i + \lambda_i) + \lambda_2^i z_2(s + \lambda_i + \lambda_i)\right] \frac{1}{A(s)}$$

(25)

Further, the L.T. of the probability that the system is in failed state at time $t$ is given by

$$\tilde{P}_{down}(s) = \left[\lambda_i^i \lambda_2^i Z_i(s) + \lambda_i \lambda_2 z_1(s + \lambda_i + \lambda_i)\right] \frac{1}{A(s)}$$

(26)

Where

$$Z_i(s) = \frac{1 - \tilde{S}_i(s)}{s}$$

$$A(s) = \frac{\tilde{S}_i(s)}{s} + \lambda_i \tilde{S}_i(s) + \lambda_2^i \tilde{S}_2(s) + \lambda_i \tilde{Z}_i(s) + \lambda_2 \tilde{Z}_2(s)$$

(27)

7. Steady State Behavior of the System

Using Abel’s Lemma; viz. $\lim_{t \to \infty} P(t) = \lim_{S \to 0} \tilde{P}(S)$, we get the following steady state probabilities of the system:
\[ P_{0,0} = \frac{1}{A'(0)} \]  

Where

\[ A'(0) = \frac{d}{ds}[A(s)]|_{s=0} P_{f,0} = \lambda z_1 (\lambda_c + \lambda) \frac{1}{A(0)} \]  

\[ P_{0,f} = \lambda z_2 (\lambda_c + \lambda) \frac{1}{A'(0)} \]  

\[ P_{f,f} = [\lambda z_1 (\lambda_c + \lambda) + \lambda z_2 (\lambda_c + \lambda)] \times \frac{M_c}{A'(0)} \]  

\[ P_{c,c} = [\lambda z_1 (\lambda_c + \lambda) + \lambda z_2 (\lambda_c + \lambda)] \times \frac{M_c}{A'(0)} \]  

Also up and down state probabilities of the system are given by

\[ P_{up} = \left[ 1 + z_1 (\lambda_2 + \lambda_{c,1}) + \lambda_2 z_2 (\lambda_1 + \lambda_{c,1}) \right] \frac{1}{A'(0)} \]  

\[ P_{down} = \left[ \lambda z_1 (\lambda_2 + \lambda_{c,1}) + \lambda z_2 (\lambda_1 + \lambda_{c,1}) \right] \times \frac{1}{A'(0)} \]  

8. Particular Case

When repair follows exponential time distribution

Setting, \( S_i = \frac{\mu_i}{s + \mu_i} \) (i = 1, 2, 3, c) in equations (25) and (26), one may easily obtain:-

\[ P_{up}(s) = \left[ 1 + \frac{\lambda_1}{s + \lambda_2 + \lambda_{c,1} + \mu_1} + \frac{\lambda_1}{s + \lambda_1 + \lambda_{c,2} + \mu_2} \right] \times \frac{1}{A_1(s)} \]  

\[ P_{down}(s) = \left[ \frac{\lambda_1}{s + \mu_1} + \left\{ \frac{\lambda_2}{s + \mu_3} + \frac{\lambda_3}{s + \mu_1} \right\} \frac{\lambda_4}{s + \lambda_2 + \lambda_{c,3} + \mu_3} \right] \times \frac{1}{A_1(s)} \]
Where

\[ A_i(s) = s + \lambda_1 + \lambda_2 + \lambda_{c1} - \frac{\lambda_1 \mu_1}{s + \lambda_2 + \lambda_{c1} + \mu_1} - \frac{\lambda_2 \mu_2}{s + \lambda_1 + \lambda_{c2} + \mu_2} \]

\[ - \left[ \frac{\lambda_1 \lambda_2}{s + \lambda_2 + \lambda_{c1} + \mu_1} + \frac{\lambda_1 \lambda_2}{s + \lambda_1 + \lambda_{c2} + \mu_2} \right] \frac{\mu_3}{s + \mu_3} \]

\[ - \left[ \frac{\lambda_1 \lambda_2}{s + \lambda_2 + \lambda_{c1} + \mu_1} + \frac{\lambda_1 \lambda_2}{s + \lambda_1 + \lambda_{c2} + \mu_2} \right] \frac{\mu_c}{s + \mu_c} \]

Laplace-transform of the reliability of the system is then given by

\[ R(s) = \frac{1}{s + \lambda_1 + \lambda_2 + \lambda_{c1}} + \frac{\lambda_1}{(s + \lambda_2 + \lambda_{c1})(s + \lambda_1 + \lambda_{c2} + \mu_2)} + \frac{\lambda_2}{(s + \lambda_1 + \lambda_{c2})(s + \lambda_1 + \lambda_2 + \lambda_{c1})} \]

(37)

Meantime to failure (M.T.T.F) of the system is given by

\[ \text{M.T.T.F} = \frac{1}{\lambda_1 + \lambda_2 + \lambda_{c1}} \left[ 1 + \frac{\lambda_1}{\lambda_2 + \lambda_{c1}} + \frac{\lambda_2}{\lambda_1 + \lambda_{c2}} \right] \]

(38)

Also, variance of the time to failure is given by

\[ \sigma^2 = \left[ \lambda_2 + \lambda_{c1} \right]^2 (\lambda_1 + \lambda_{c2})^2 - \left[ (\lambda_1 + \lambda_{c1}) \lambda_1 + (\lambda_2 + \lambda_{c2}) \lambda_2 \right]^2 + 2 \left[ \lambda_1 + \lambda_{c2} \right] \lambda_1 + (\lambda_2 + \lambda_{c1})^2 \left[ \lambda_1 + \lambda_2 + \lambda_{c1} \right] \left[ \lambda_1 + \lambda_2 + \lambda_{c1} \right] \]

(39)

After taking inverse L.T., equation (37) yields the reliability of the system at time t; viz.

\[ R(t) = \exp \left[ - \left( \lambda_1 + \lambda_2 + \lambda_{c1} \right) t \right] + \left[ \lambda_1 + \lambda_2 + \lambda_{c1} \right] \exp \left[ - \left( \lambda_1 + \lambda_2 + \lambda_{c1} \right) t \right] \]

(40)

9. Numerical Computation

Setting some suitable value of \( \lambda_1, \lambda_2, \lambda_{c1}, \lambda_{c2} \)

\( \lambda_{c1} \) in equation (38) and of \( \lambda_1, \lambda_2, \lambda_{c1}, \lambda_{c2} \) in equation (40), one may sketch the graphs;

M.T.T.F.

V.N. and reliability V.time respectively, which are shown in the adjoining figures.

Table 1

<table>
<thead>
<tr>
<th>( \lambda_{c1} )</th>
<th>0.01</th>
<th>0.02</th>
<th>0.03</th>
</tr>
</thead>
</table>
Table 1

<table>
<thead>
<tr>
<th>No.</th>
<th>$\lambda_0$</th>
<th>M.T.T.F</th>
<th>$\lambda_2 = \lambda_3 = 0$</th>
<th>$\lambda_2 = \lambda_3 = 0.0005$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.002</td>
<td>72.91</td>
<td>71.98</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.004</td>
<td>68.62</td>
<td>67.74</td>
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</tr>
<tr>
<td>3</td>
<td>0.006</td>
<td>64.81</td>
<td>63.98</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.008</td>
<td>61.40</td>
<td>60.61</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.010</td>
<td>58.33</td>
<td>57.58</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.012</td>
<td>55.55</td>
<td>54.84</td>
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</tr>
<tr>
<td>7</td>
<td>0.014</td>
<td>53.03</td>
<td>52.35</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.016</td>
<td>50.72</td>
<td>50.07</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0.018</td>
<td>48.61</td>
<td>47.98</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.020</td>
<td>46.66</td>
<td>46.06</td>
<td></td>
</tr>
</tbody>
</table>
**Table 2**

\( \lambda_3 = 0.01, \lambda_2 = \lambda_1 = 0.02, \lambda_3 = 0.03 \)

<table>
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<th>No.</th>
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<th>R(t)</th>
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<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.9995</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
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</tr>
<tr>
<td>4</td>
<td>3</td>
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</tr>
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</tr>
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</tr>
<tr>
<td>11</td>
<td>10</td>
<td>0.9706</td>
</tr>
</tbody>
</table>
M.T.T.F/ $\lambda_{c_1}$

![Graph showing M.T.T.F/ $\lambda_{c_1}$](image)

- $\lambda_{c_2} = \lambda_{c_3} = 0$
- $\lambda_{c_1} = \lambda_{c_3} = 0.0005$
- $\lambda_1 = 0.01, \lambda_2 = 0.02, \lambda_3 = 0.03$

**FIG. 2**
\[ \lambda C_2 = \lambda C_3 = 0 \]

\[ \lambda C_1 = 0.005, \quad \lambda C_2 = 0.006, \quad \lambda C_3 = 0.007 \]

\[ \lambda_1 = 0.01, \quad \lambda_1' = \lambda_2 = 0.02, \quad \lambda_2 = 0.03 \]

![Figure 3](image)

**References**


